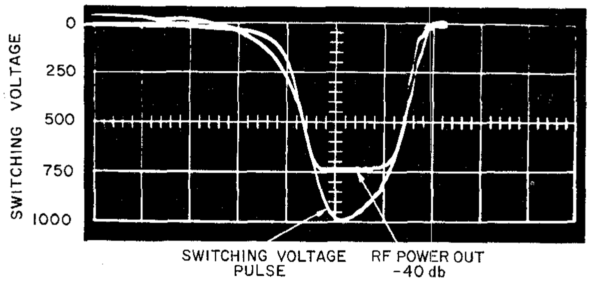

Fig. 8. Oscilloscope photograph of dc switching pulse and RF output pulse.

From these tests, it is reasonable to say that the switching time is less than 100 ns.

Power Handling Capability

Average and peak power levels at various pulse widths were used to check the power handling capability of the ferroelectric switch.

It was found that the maximum average power the switch could handle and still exhibit the above quoted characteristics was approximately 500 milliwatts. This value changes slightly with peak power and pulse width. At an average power level of 500 mW the unit starts to limit and reflection occurs. No apparent damage occurred up to a maximum input power of 5 watts.

It appears feasible that the unit could be retuned to operate at higher power levels but then it would not function at reduced powers.

Acknowledgment

The authors gratefully acknowledge the assistance of John Agrios and the Computer Laboratory personnel at USAEL in providing the numerical data related to the computer program portion of this study. The authors also wish to acknowledge the assistance of Karl Schmidt who prepared many of the curves from the computer data, and Roger Denhard for assistance in carrying out the measurements.

Bibliography

[1] W. J. Gemulla and R. D. Hall, "Ferroelectrics at microwave frequencies," *Microwave J.*, vol. 3, pp. 47–51, February 1960.
[2] C. P. Sharpe and C. G. Brochus, "Investigation of microwave properties of ferroelectric materials (final report)," University of Michigan Research Inst., Ann Arbor, Michigan, Rept. 2732-4-F, March 1959.
[3] M. Cohn and A. F. Eikenburg, "Ferroelectric phase shifters for VHF and UHF," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-10, pp. 536–548, November 1962.
[4] G. Ruprecht, B. D. Silverman, and R. J. Bell, "Investigation of the microwave properties of ferroelectrics," AFCRC Labs., Bedford, Mass., Rept. AF 19(604)-4085, June 16, 1958, October 15, 1960.

Computation of Impedance and Attenuation of TEM-Lines by Finite Difference Methods

M. V. SCHNEIDER, MEMBER, IEEE

Abstract—The characteristic impedance and the attenuation of transmission lines supporting TEM modes can be computed by using finite difference methods for solving the Laplace equation for the domain defined by the inner and the outer conductor. The difference equations can be solved by machine computation and the impedance and the attenuation is obtained by integrating the field gradients and the squares of field gradients over both boundaries.

The case of a shielded strip transmission line is treated as a numerical example. A computation time of approximately 0.015 hour on the IBM 7094 is required for achieving an accuracy of 0.5 percent for the impedance and 2 percent for the attenuation.

The finite difference method is also used for lines which are partially filled with dielectric material and it is concluded that low attenuations are obtained by placing the dielectric material in such a way that high field regions are avoided.

Manuscript received May 14, 1965; revised August 9, 1965.
The author is with Bell Telephone Laboratories, Inc., Holmdel, N. J.

Introduction

The COMPUTATION of the characteristic impedance and the attenuation of various transmission lines supporting TEM modes is a problem of considerable importance for the design of microwave circuits. The impedance and the attenuation of such lines can be computed by using conformal transformation techniques. Various dictionaries and lists of conformal transformations covering a large number of cases have been published by Moon and Spencer [1], Kober [2], and Binns and Lawrenson [3], however, only a limited number of the transformations are applicable to transmission lines which occur in practice. It is, therefore, understandable that considerable work has been spent on numerical techniques for computing the characteristic impedance of several transmission lines.

A few recent examples are the variational method by Duncan [4], the orthonormal block analysis used by Cruzan and Garver [5], and the approximate solution of an appropriate integral equation by Cristal [6].

The purpose of this paper is to show that finite difference techniques are particularly suited for the evaluation of the characteristic impedance as well as the attenuation of transmission lines by machine computation. The technique consists basically of a method for solving the field equations by replacing the domain between the conductors by a finite set of points called mesh points and by solving the Laplace equation in finite difference form by digital computation.[1] The technique can be extended, with certain restrictions, to transmission lines which are partially filled with dielectric material. It is furthermore possible to use similar methods for systems of partial differential equations and for two- or three-dimensional eigenvalue equations which makes it possible to find solutions for guided wave structures as well as electromagnetic resonators. Problems of this type have been reviewed by Alder [7] and Forsythe and Wasow [8] and they will not be treated in the following sections.

Finite Difference Equation for the Laplace Operator

Let us assume that the cross section of the transmission line is defined by the boundaries 1 and 2 shown in Fig. 1. The potential function $U(x, y)$ may be found by solving the Laplace equation for the domain defined by the boundaries 1 and 2. The function $U(x, y)$ satisfies the linear second-order partial differential equation

$$U_{xx} + U_{yy} = 0 \tag{1}$$

with the following boundary conditions

$$U(x, y) = U_1 = 1 \qquad \text{(boundary 1)} \tag{2}$$

$$U(x, y) = U_0 = 0 \qquad \text{(boundary 2).} \tag{3}$$

The problem can be simplified if the structure has a line of symmetry. The domain can be reduced to a subdomain shown in Fig. 1, with the additional boundary conditions

$$\frac{\partial U}{\partial n} = 0 \qquad \text{(boundaries 3 and 4).} \tag{4}$$

The problem is thus transformed into a combination of the classical Dirichlet and the Neumann problem since the normal derivative $\partial U/\partial n$ is specified on boundaries 3 and 4.

A square mesh with an arbitrary mesh size h is now superimposed on the subdomain. By using the notation of Fig. 2 and assuming that $U(x, y)$ has partial derivatives of fourth order in the neighborhood around the interior mesh point (x_0, y_0), we obtain with Taylor's theorem

$$U_{xx}(x_0, y_0) = \frac{U(x_0 + h, y_0) + U(x_0 - h, y_0) - 2U(x_0, y_0)}{h^2} - \frac{h^2}{4!}\left[U_{xxxx}(\xi_1, y_0) + U_{xxxx}(\xi_2, y_0)\right] \tag{5}$$

$$U_{yy}(x_0, y_0) = \frac{U(x_0, y_0 + h) + U(x_0, y_0 - h) - 2U(x_0, y_0)}{h^2} - \frac{h^2}{4!}\left[U_{yyyy}(x_0, \eta_1) + U_{yyyy}(x_0, \eta_2)\right]. \tag{6}$$

The coordinates ξ_0, ξ_2, η_1, and η_2 satisfy the following conditions

$$x_0 - h < \xi_1 < x_0 < \xi_2 < x_0 + h \tag{7}$$

$$y_0 - h < \eta_1 < y_0 < \eta_2 < y_0 + h. \tag{8}$$

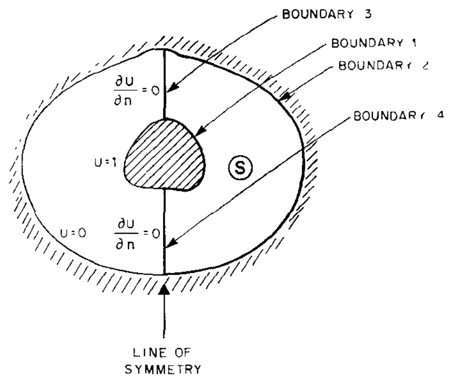

Fig. 1. Cross section of transmission line with inner and outer conductor.

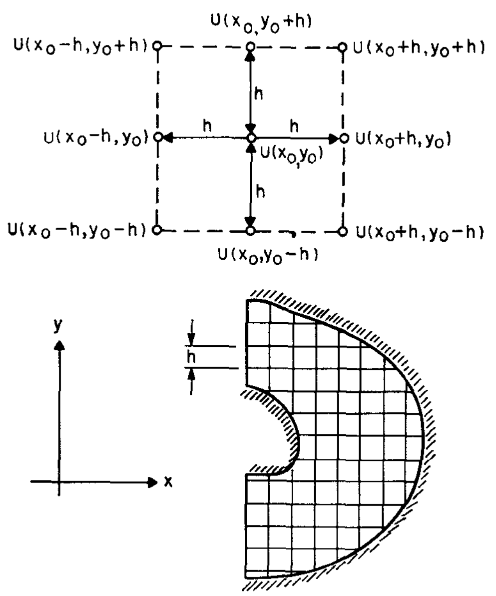

Fig. 2. Superposition of square mesh on subdomain S and notation used for potential U for adjacent mesh points.

[1] Finite difference techniques for solving boundary value problems are also used in a paper by H. E. Green, "The numerical solution of some important transmission-line problems," *IEEE Trans. on Microwave Theory and Techniques*, pp. 676–692, September 1965. H. E. Green uses a method for finding the solution of a large group of simultaneous equations. The author of the present paper uses a scanning technique of the mesh by digital methods and discusses furthermore the computation of attenuation.

The combination of (1), (5), and (6) results in

$$U(x_0+h, y_0) + U(x_0-h, y_0) + U(x_0, y_0+h) + U(x_0, y_0-h) - 4U(x_0, y_0) = \delta(h^4) \quad (9)$$

where $\delta(h^4)$ is an error term of fourth order. This relationship can be represented by the following symbolic notation

$$\begin{Bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{Bmatrix} \begin{Bmatrix} U(x_0-h, y_0+h) & U(x_0, y_0+h) & U(x_0+h, y_0+h) \\ U(x_0-h, y_0) & U(x_0, y_0) & U(x_0+h, y_0) \\ U(x_0-h, y_0-h) & U(x_0, y_0-h) & U(x_0+h, y_0-h) \end{Bmatrix} = \delta(h^4). \quad (10)$$

The elements of the first matrix represent the coefficients corresponding to a subset of nine adjacent mesh points, and the multiplication of the two matrices is *defined* by

$$A \cdot B = \sum_{i,j} a_{ij} \cdot b_{ij}. \quad (11)$$

It is of course possible to write the same equation by considering only points along the diagonals. However, one has to keep in mind that the lattice spacing h has to be replaced by the spacing between diagonal elements

$$h_d = \sqrt{2}\, h \quad (12)$$

and one obtains

$$\begin{Bmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 1 \end{Bmatrix} \cdot U = \delta(h_d^4). \quad (13)$$

Equations (10) and (13) may be combined into a single expression. The weight of the coefficients for diagonal elements and for both vertical and horizontal elements has to be different since the error term is a function of h^4. Since,

$$h_d^4 = 4h^4 \quad (14)$$

four times less weight must be attached to the diagonal elements. A more detailed analysis shows that the new error term obtained by combining (1) and (13) with suitable weight factors is $\delta(h^6)$. The result is thus

$$\begin{Bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{Bmatrix} \cdot U = \delta(h^6). \quad (15)$$

The first matrix can be regarded as a Laplace Operator written in finite difference form, and (15) becomes

$$L \cdot U = \delta(h^6). \quad (16)$$

The error term can be further reduced by including more and more mesh points. It must be remembered, however, that other types of errors are introduced if the inner boundary has reentrant corners because $U(x, y)$ does not have continuous partial derivatives when the point (x, y) is coincident with an edge or corner point.

The Relaxation Process

A review of the more common methods for solving (10) or similar equations of higher order on a two- or three-dimensional domain has been given by Frankel [9], Southwell [10], and more recently by Forsythe and Wasow [8]. The conclusion is that the most effective methods are still the numerical procedures originally proposed between 1908 and 1928 by Runge [11], Richardson [12], and Courant [13]. The underlying idea for finding a numerical solution is to approximate the function $U(x, y)$ with a function $U_{j,k}$ which is only defined for each discrete mesh point. $U_{j,k}$ satisfies (10), (15), (16) or similar equations with a vanishing error term

$$L \cdot U_{j,k} = 0. \quad (17)$$

An initial, or guess, value $U_{j,k}^0$ is first assigned to each interior mesh point of the domain. Successive approximations for $U_{j,k}$ are obtained from

$$U_{j,k}^{n+1} = U_{j,k}^n + \alpha L \cdot U_{j,k}^n. \quad (18)$$

The relaxation factor α determines the rate of convergence. From (10) and for $\alpha = 1/4$ comes the familiar Richardson equation

$$U_{j,k}^{n+1} = \tfrac{1}{4}(U_{j-1,k}^n + U_{j+1,k}^n + U_{j,k-1}^n + U_{j,k+1}^n). \quad (19)$$

Successive scanning of the lattice by a systematic procedure or by a random process will lead to a series of numbers $U_{j,k}^n$ which will hopefully converge into $U_{j,k}$.

Equations (18) and (19) are not particularly suitable for digital computing since two successive values $U_{j,k}^n$ and $U_{j,k}^{n+1}$ have to be retained in core storage at the same time. A more suitable numerical procedure is the Liebmann method [11]. In its simplest form, the lattice is scanned along successive rows, and old values for each mesh point are discarded and replaced by new ones. The relaxation formula for the new value of $U_{j,k}^{n+1}$ is

$$U_{j,k}^{n+1} = \alpha(U_{j-1,k}^{n+1} + U_{j+1,k}^n + U_{j,k-1}^{n+1} + U_{j,k+1}^n) - (4\alpha - 1)U_{j,k}^n \quad (20)$$

or, with $\alpha = 1/4$

$$U_{j,k}^{n+1} = \tfrac{1}{4}(U_{j-1,k}^{n+1} + U_{j+1,k}^n + U_{j,k-1}^{n+1} + U_{j,k+1}^n). \quad (21)$$

For later reference it is more convenient to write (20) in the form

$$U_{j,k}^{n+1} = \omega \frac{U_{j-1,k}^{n+1} + U_{j+1,k}^{n} + U_{j,k-1}^{n+1} + U_{j,k+1}^{n}}{4} - (\omega - 1) U_{j,k}^{n} \tag{22}$$

with

$$\omega = 4\alpha. \tag{23}$$

Organization of the Program

The digital computer program for calculating the attenuation and the characteristic impedance consists of a sequence of open subroutines.

The following open subroutines are used:

1) Location Loader
2) Relaxtion Process
3) Subdivision into Finer Mesh
4) Looping between Steps 1) and 2).

A. Location Loader

It has been pointed out previously that an initial or guess value $U_{j,k}^0$ must be assigned to each mesh point. Fixed potentials are assigned to the inner and the outer boundary and intermediate values between these two fixed potentials are assigned to the mesh points covering the subdomain. This procedure is illustrated for the case of a shielded strip transmission line[2] shown in Fig. 3. The subdomain between the inner and the outer conductor is covered by a square mesh with mesh size h shown in Fig. 4. It is convenient to choose the mesh size in such a way that grid lines coincide with the boundaries and the two dotted mirror lines in Fig. 3. This requires that the ratios w/b, t/b, and s/b are rational numbers.

A constant potential $U_1 = 1$ is assigned to all mesh points on the inner boundary and $U_0 = 0$ to all mesh points on the outer boundary. An initial or guess value $U_{i,j}^0$ is assigned to all interior mesh points. This initial value is

$$U_{i,j}^0 = \frac{m + l - i - 1}{m - 1} \qquad 1 \leqq j \leqq n \tag{24}$$

$$U_{i,j}^0 = \frac{(n + k - j - 1) \cdot U_{i,n}^0}{k - 1} \qquad n + 1 \leqq j \leqq n + k - 2 \tag{25}$$

with the notation shown in Fig. 4. This particular choice insures a linear potential drop on all grid lines connecting the inner and the outer boundary.

B. Relaxation Process

Equations (21) and (22) are used for the relaxation process. The procedure starts on the inner boundary and moves through successive rows from right to left. The process is reversed after reaching the last point on the subdomain. The whole net is completely scanned several hundred times before it is subdivided into a finer mesh with

$$h_{\text{NEW}} = \tfrac{1}{2} h_{\text{OLD}}. \tag{26}$$

It was found, as expected, that the convergence rate can be improved drastically by using an optimum value for ω in (22). This does not necessarily lead to a reduction in total computation time since

1) The optimum value for ω is not known theoretically and has to be found by trial and error.
2) Equation (22) requires more instructions in machine language than (21) for relaxing a single point. An additional point to consider is that (22) requires two floating-point multiplications by ω and by $(\omega - 1)$ with a large number of machine cyles. On the other hand only few machine cycles are necessary in (21) for dividing by four because division by four can be replaced in machine language by a fixed-point subtraction of a binary ten from the characteristic part of the machine word.

Figure 5 shows the value of the potential U at a fixed point P for successive relaxations as a function of ω. The line configuration is the same as shown in Fig. 4. Oscillations occur for sufficiently large values of ω and it can be shown that the process becomes unstable for $\omega \geqq 2$.

C. Subdivision into Finer Mesh

The subdivision into a finer mesh is shown symbolically in Fig. 4. The refinement of the net is made by

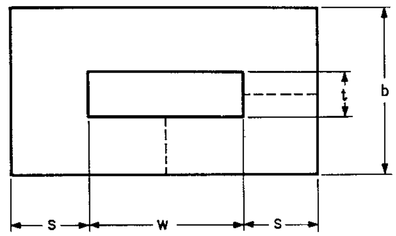

Fig. 3. Shielded strip transmission line with dotted mirror lines.

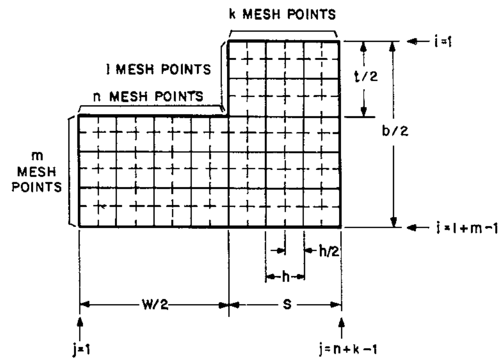

Fig. 4. Subdivision of subdomain into mesh.

[2] This type of line cannot be treated by exact conformal mapping. It can be analysed by various conformal mapping approximations and other numerical methods as shown by Getsinger [14], Izatt [15], and Joines [16].

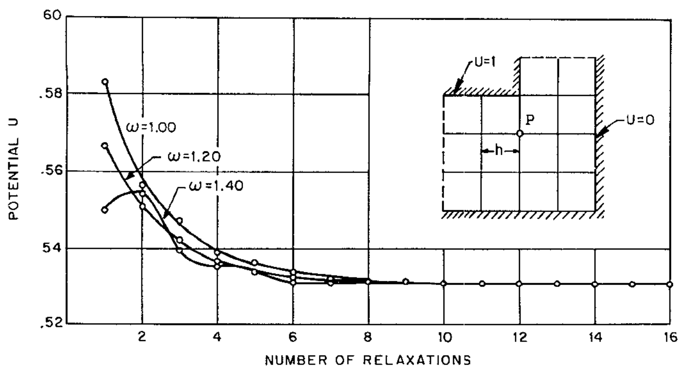

Fig. 5. Potential U for point P for successive relaxations with $\omega=1.0$, 1.2, and 1.4.

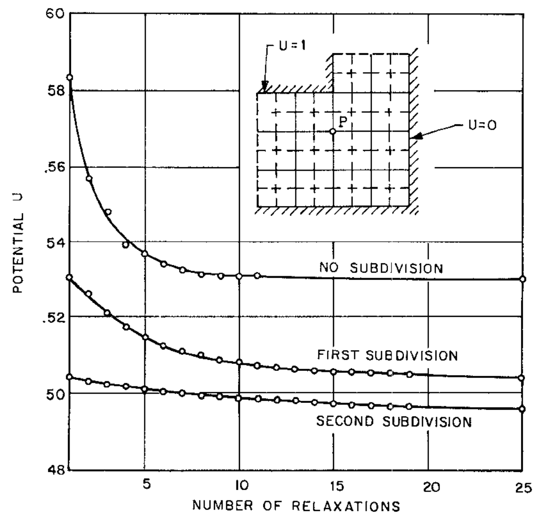

Fig. 6. Potential U for point P for successive relaxations with subdivisions.

assigning new locations to the old mesh points in core storage and by storing the average of the four closest neighboring points into each storage location for a new point. It is self evident that special precautions are necessary on mirror lines and for mesh points adjacent to curved boundaries.

Figure 6 shows the value of the potential U at the fixed point P for successive relaxations and for successive subdivisions. The value of ω is equal to one. The potential reaches a different asymptotic value after each subdivision. This is expected since the error term is a function of the mesh size h.

D. Looping Between Subdivision of the Mesh and Relaxation Process

The relaxation process is resumed after the subdivision of the mesh and it is continued until the difference

$$\Delta_{j,k}^{n} = U_{j,k}^{n+1} - U_{j,k}^{n} \tag{27}$$

for certain test points becomes sufficiently small. The subdivision and the relaxation routine are repeated until a sufficient number of mesh points are available close to the boundaries to obtain the characteristic impedance and the attenuation of the transmission line.

Computation of Impedance and Attenuation

The characteristic impedance Z and the attenuation α of a transmission line supporting a TEM mode can be expressed in terms of line integrals of the electric field vector E_n normal to the inner boundary S1 and the outer boundary S2

$$Z = \sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{V}{\oint_{S1} E_n ds} \tag{28}$$

$$\alpha = \frac{R_M}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{\oint_{S1+S2} E_n^2 ds}{\oint_{S1} E_n ds} \tag{29}$$

with

$R_M = 1/\sigma\delta$ Surface resistance
$\sqrt{\mu_0/\epsilon_0} = 120\pi$ (Ohm)
σ = Conductivity of wall material
δ = Skin depth
V = Voltage between inner and outer conductor
ds = Line differential on inner boundary S1 or on outer boundary S2.

It has to be pointed out that a free space impedance of 120π is obtained for $c=3\cdot10^6$ m/s. It has also been assumed that the skin depth is small with respect to the line dimensions.

Values for the electric field vector E_n on both boundaries are obtained by calculating the difference of the potential U for two appropriate mesh points located adjacent to the boundary. The line integrals are evaluated by a simple summation over both boundaries.

Equation (29) shows that the surface resistance R_M and a unit length L of the transmission line have to be specified in order to compute the attenuation. It is convenient to put $R_M=1$ ohm and to choose one characteristic dimension of the cross section of the line as unit length L. In the case of the transmission line shown in Fig. 3, we choose $L=b$ and we define the normalized attenuation α_N as

$$\alpha_N = \frac{b\cdot\alpha}{R_M}\ \text{(neper/ohm)}. \tag{30}$$

All subsequent results will be given in terms of the normalized attenuation α_N.

The results of a series of computations for a shielded strip transmission line are shown in Figs. 7 and 8 for three fixed ratios $t/b=0.1$, 0.2, and 0.3 as a function of s/b. The ratio w/b is constant for all cases and is equal to 0.8. This particular choice was made in order to obtain lines with a characteristic impedance in the vicinity of 50 ohms. The important parameters involved in the numerical evaluation of a single point are given in Table I.

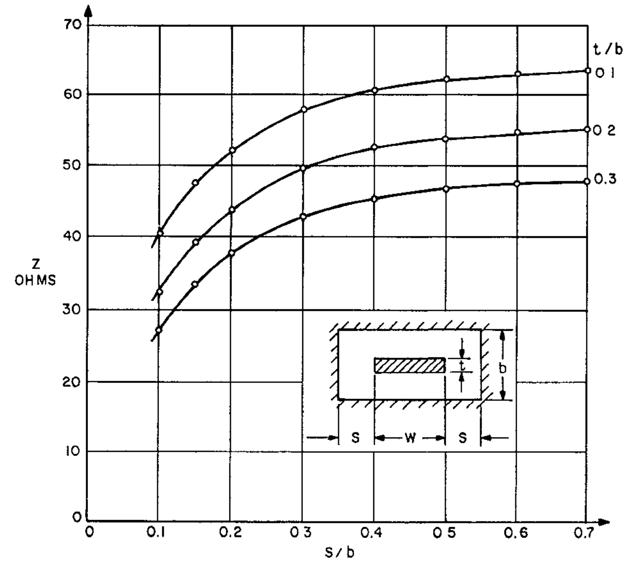

Fig. 7. Characteristic impedance of shielded strip transmission line as a function of s/b with $w/b=0.8$.

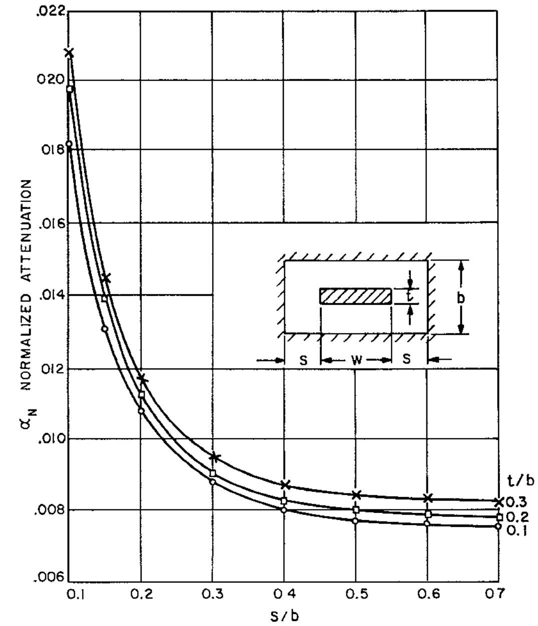

Fig. 8. Normalized attenuation of shielded strip transmission line in neper/ohm as a function of s/b with $w/b=0.8$.

TABLE I

NUMERICAL PARAMETERS FOR COMPUTATION OF IMPEDANCE AND ATTENUATION FOR $t/b=0.2$, $s/b=0.5$, $w/b=0.8$

Strip Transmission Line	Initial Mesh	First Subdivision	Second Subdivision
Total number of mesh points on inner conductor	40	80	160
Total number of mesh points on outer conductor	112	224	448
Number of relaxations for each mesh point	100	200	400

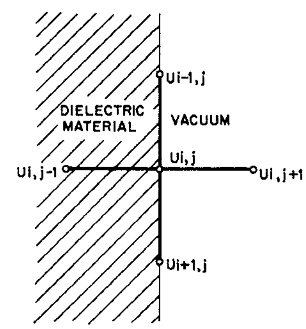

Fig. 9. Mesh points on dielectric boundary.

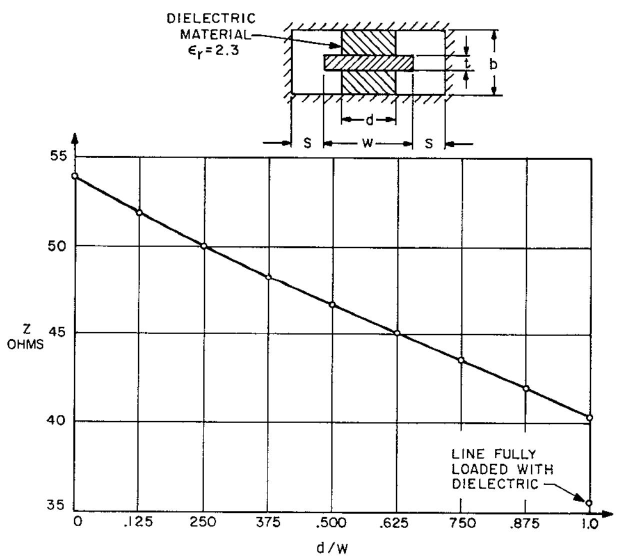

Fig. 10. Impedance of partially loaded line with $t/b=0.2$, $s/b=0.5$, and $w/b=0.8$.

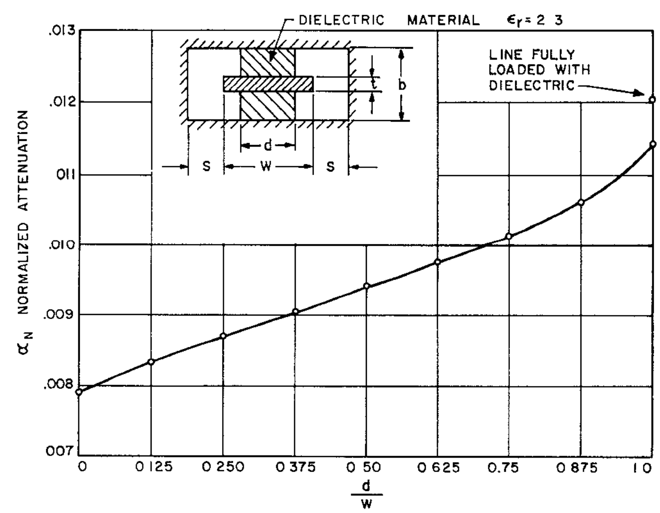

Fig. 11. Normalized attenuation of partially loaded line in neper/ohm as a function of d/w. The unit length is equal to the ground plane spacing b.

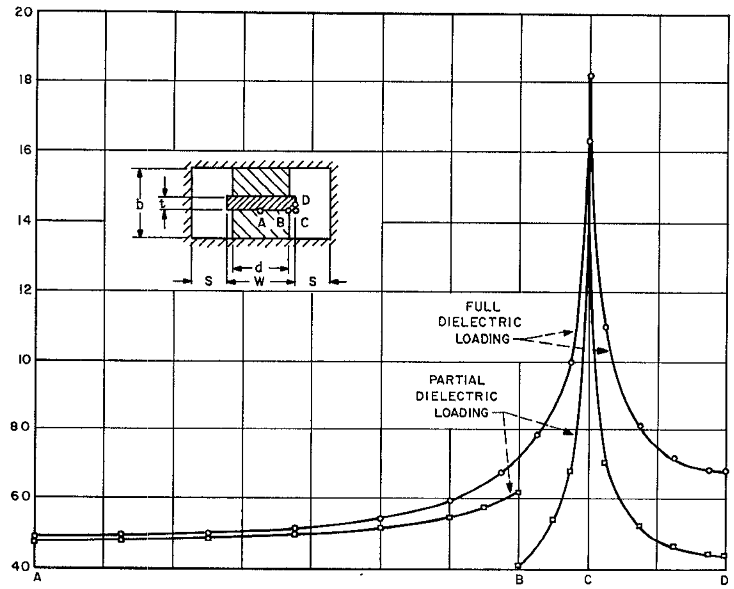

Fig. 12. Current density on inner conductor in arbitrary units for $w/b=0.8$, $s/b=0.5$, $t/b=0.2$, $d/w=0.875$, and $\epsilon_r=2.3$.

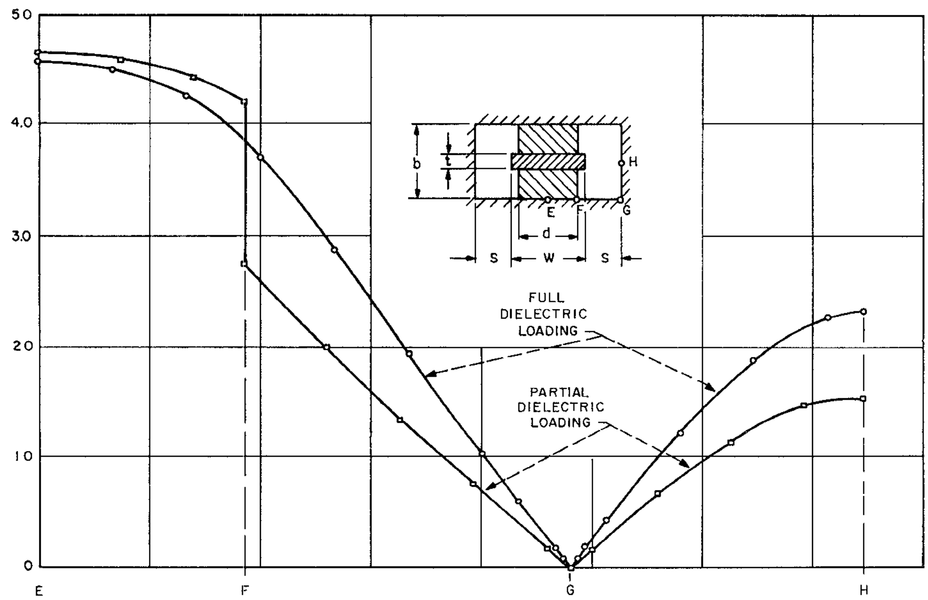

Fig. 13. Current density on outer conductor in arbitrary units for $w/b=0.8$, $s/b=0.5$, $t/b=0.2$, $d/w=0.875$, and $\epsilon_r=2.3$.

The accuracy of the impedance calculation can be estimated by using the fact that the error term discussed in (9) is a function of the mesh spacing h. A further check is obtained by comparing separately the line integrals of the field gradients for the inner and the outer boundary. The two integrals lead to an upper and a lower bound for the characteristic impedance. The accuracy of the impedance calculation is 0.5 percent. The accuracy achieved for the computation of the attenuation is lower because of increased errors due to the high fields at the reentrant corners on the inner boundary. This accuracy is estimated at 2 percent. The total machine time required for the evaluation of a single point (impedance and attenuation) on the IBM 7094 is approximately 0.015 hour. Higher accuracies may be achieved by further subdivisions of the mesh and a further increase of the number of relaxations for each mesh point. It was found, however, that the machine time required for such a procedure becomes excessive and that there are some additional problems because of the limited number of available locations in core storage.

The Strip Transmission Line with Partial or Full Dielectric Filling

For practical purposes it is necessary to support the center conductor of a strip transmission line. This can be accomplished by filling the space between the conductors fully or partially with dielectric material. The case of a completely filled line with a dielectric material having a dielectric constant ϵ_r is analytically simple because the impedance will decrease by a factor $\sqrt{\epsilon_r}$ and the attenuation increases by a factor $\sqrt{\epsilon_r}$. This increase is further enhanced by the dielectric losses and it is therefore advisable to reduce the amount of dielectric material in order to obtain small total losses.

Several new problems, however, arise in the case of a partially filled line because such a line will not support a TEM mode which means that a characteristic impedance of the line cannot be defined. It is nevertheless possible to obtain good approximations if the line dimensions are much smaller than half a wavelength which means that the operating frequency is far below cutoff for all higher order modes. Similar assumptions have been made in a recent paper by Wheeler [17] for the special case of a transmission line of parallel strips separated by a dielectric sheet.

The following results are all restricted to this static case. This means that (21) or (22) is applied for the whole mesh except for all mesh points located on the interface between the two regions $\epsilon_r \neq 1$ and $\epsilon_r = 1$. The difference equation for these points can be derived by requiring that the normal components of the displacement are continuous at the interface and that the finite difference equation has to be identical with (3) for $\epsilon_r = 1$. The result is

$$U_{i,j} = \frac{U_{i,j+1} + \epsilon_r U_{i,j-1}}{2(1 + \epsilon_r)} + \frac{U_{i-1,j} + U_{i+1,j}}{4} \qquad (31)$$

with the notation shown in Fig. 9.

The results for a partially loaded strip transmission line are shown in Figs. 10 and 11. A choice of line parameters had to be made and it was again decided to treat a case with an impedance in the vicinity of 50 ohms. It was furthermore decided to use a dielectric constant $\epsilon_r = 2.3$ since most dielectric materials used in practical circuits fall into the range $\epsilon_r = 2.2$ to 2.4. Any other case can be treated by simply changing the input parameters of the computer program.

A linear decrease of the impedance occurs in Fig. 10 if the ratio d/w is gradually increased. This behavior is expected since the structure resembles a parallel plate capacitor and since the effects from the fringe fields including the shielding sidewalls are small. Figure 11 shows that the attenuation behaves differently. The attenuation increases nearly linear for low values d/w and then more rapidly as the boundary of the dielectric material approaches the reentrant corner ($d/w = 1$) on the center conductor.

This behavior can be explained as follows. The current density on the reentrant corner is greatly enhanced if the dielectric material reaches the corner or if the corner is completely embedded by the material. This means that dielectric supports should be designed in such a way that they are far from any corner on the inner conductor. The situation for the outer conductor is different, that means the supports should extend into the corners of the outer conductor in order to get higher current densities in the low field regions.

Figures 12 and 13 substantiate the previous statements. Figure 12 is a current density plot for the inner conductor with $d/w = 0.875$. A comparison with the current density for full dielectric loading shows that the density is reduced in the corner region C because the dielectric material does not extend into this region. Figure 13 shows the current distribution on the outer conductor. The current density at point E is slightly smaller than the density at point A on the inner conductor. The partial dielectric loading reduces the density in the corner region G and reduces also the effect of the shielding side wall.

Conclusions

The relaxation process is a simple and accurate technique for computing the impedance and the attenuation of transmission lines supporting TEM modes. It is particularly useful for the special case of a shielded strip transmission line because exact solutions are not known and good approximations are only available for limited ranges of the line parameters.

The case of transmission lines with dielectric beads or supports can be treated by the same technique with certain restrictions. It is found that the dielectric should be placed into low field regions of the transmission line in order to reduce ohmic losses.

References

[1] P. Moon and D. E. Spencer, *Field Theory Handbook*. Göttingen: Springer, 1961.

[2] H. Kober, *Dictionary of Conformal Representations*. New York: Dover, 1957.

[3] K. J. Binns, andand P. J. Lawrenson, *Analysis and Computation of Electric and Magnetic Field Problems*. New York: MacMillan, 1963.

[4] J. W. Duncan, Characteristic impedance of multiconductor strip transmission lines," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-13, pp. 107–118, January 1965.

[5] O. R. Cruzan, and R. V. Garver, "Characteristic impedance of rectangular coaxial transmission lines," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-12, pp. 488–495, September 1964.

[6] E. G. Cristal, "Coupled circular cylindrical rods between parallel ground planes," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-12, pp. 428–439, July 1964.

[7] B. Alder, S. Fernbach, and M. Rotenberg, *Methods in Computational Physics*. vol. 3, New York: Academic, 1964.

[8] G. E. Forsythe, and W. R. Wasow, *Finite-Difference Methods for Partial Differential Equations*. New York: Wiley, 1960.

[9] S. P. Frankel, "Convergence rates of iterative treatments of partial differential equations," *Mathematical Tables and Other Aids to Computation*, vol. 4, pp. 65–75, April 1950.

[10] R. V. Southwell, *Relaxation Methods in Theoretical Physics*. New York: Oxford, vol. 1, 1946, vol. 2, 1956.

[11] C. Runge, "Ueber eine Methode die partielle Differentialgleichung Δu = constant numerisch zu integrieren," *Math. Phys.*, vol. 56, pp. 225–233, 1908.

[12] L. R. Richardson, "Approximate arithmetic solution by finite differences of physical problems involving differential equations with an application to the stresses in a masonry dam," *Phil. Trans. (GB)*, vol. 210A, pp. 307–357, 1910.

[13] R. Courant, "Ueber die partiellen Differentialgleichungen der mathematischen Physik," *Math. Ann.*, vol. 100, pp. 32–72, 1928.

[14] W. J. Getsinger, "Coupled rectangular bars between parallel plates," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-10, pp. 65–72, January 1962.

[15] J. B. Izatt, "Characteristic impedance of two special forms of transmission line," *Proc. IEE (GB)*, vol. 111, pp. 1551–1555, September 1964.

[16] W. T. Joines, "The characteristic impedance of symmetrical strip transmission lines with undesired modes suppressed," Ph.D. thesis, Department of Electrical Engineering, Duke University, Durham, N. C., May 1964, to be published.

[17] H. A. Wheeler, "Transmission-line properties of parallel strips separated by a dielectric sheet," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-13, pp. 172–185, March 1965.